

Applications of the Ashtekar gravity to four dimensional hyperkähler geometry and Yang-Mills instantons

¶*Yoshitake HASHIMOTO , †Yukinori YASUI ,
‡Sayuri MIYAGI and §Takayoshi OTSUKA

¶Department of Mathematics, Osaka City University, Sumiyoshiku, Osaka, Japan
Department of Physics, Osaka City University, Sumiyoshiku, Osaka, Japan

Abstract

The Ashtekar-Mason-Newman equations are used to construct the hyperkähler metrics on four dimensional manifolds. These equations are closely related to anti self-dual Yang-Mills equations of the infinite dimensional gauge Lie algebras of all volume preserving vector fields. Several examples of hyperkähler metrics are presented through the reductions of anti self-dual connections. For any gauge group anti self-dual connections on hyperkähler manifolds are constructed using the solutions of both Nahm and Laplace equations.

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*E-mail address : hashimot@sci.osaka-cu.ac.jp

†E-mail address : yasui@sci.osaka-cu.ac.jp

‡E-mail address : miyagi@sci.osaka-cu.ac.jp

§E-mail address : ootsuka@sci.osaka-cu.ac.jp

1 Introduction

Four dimensional hyperkähler geometry has been studied extensively in connection with several issues: gravitational instantons, supersymmetric non-linear σ models and compactifications of superstrings [1, 2, 3]. A hyperkähler manifold is a Riemannian manifold equipped with three covariantly constant complex structures J^a ($a = 1, 2, 3$) which obey the condition $J^a J^b = -\delta_{ab} - \epsilon_{abc} J^c$. In four dimension, a Riemannian manifold is hyperkähler if and only if its Ricci curvature is zero and its Weyl curvature is either self-dual or anti self-dual, namely, the metric is the solution of (anti) self-dual Einstein equation.

It is known that all asymptotically locally Euclidean (ALE) hyperkähler manifolds are classified by Dynkin diagrams of ADE types. These spaces are constructed as hyperkähler quotients of flat Euclidean spaces [4].

On the other hand, in the Hamiltonian approach for general relativity, Ashtekar and Mason-Newman reduced the problem of finding hyperkähler metrics to that of finding linearly independent four vector fields V_μ ($\mu = 0, 1, 2, 3$) and a volume form ω on a four dimensional manifold X that satisfy the following two conditions [5, 6, 7]:

(1) volume preserving condition

$$L_{V_\mu} \omega = 0 \quad (1.1)$$

(2) half-flat condition

$$\frac{1}{2} \bar{\eta}_{\mu\nu}^a [V_\mu, V_\nu] = 0, \quad (1.2)$$

where $\bar{\eta}_{\mu\nu}^a$ ($a = 1, 2, 3$) are the 't Hooft matrices satisfying the relations:

$$\bar{\eta}_{\mu\nu}^a = -\bar{\eta}_{\nu\mu}^a, \quad \bar{\eta}_{\mu\nu}^a \bar{\eta}_{\nu\sigma}^b = \delta_{ab} \delta_{\nu\sigma} + \epsilon_{abc} \bar{\eta}_{\nu\sigma}^c. \quad (1.3)$$

The hyperkähler metric on X is given by $g(V_\mu, V_\nu) = \phi \delta_{\mu\nu}$ with $\phi = \omega(V_0, V_1, V_2, V_3)$ (we choose the sign of ω such that ϕ is positive.). Then three complex structures J^a are expressed by

$$J^a(V_\mu) = \bar{\eta}_{\nu\mu}^a V_\nu. \quad (1.4)$$

Conversely the pair (V_μ, ω) can be locally constructed from any hyperkähler structure (g, J^a) as follows [6, 8]: for a harmonic function τ and the volume form ω_g with respect to g , the vector fields V_μ are defined by

$$V_0 = \phi \text{grad } \tau, \quad (1.5a)$$

$$V_a = -J^a(V_0), \quad (a = 1, 2, 3), \quad (1.5b)$$

where $\phi = g(\text{grad}\tau, \text{grad}\tau)^{-1}$. Then V_μ preserve the volume form $\omega = \phi^{-1} \omega_g$ and satisfy the half-flat condition. We can choose the local coordinates such that $V_0 = \frac{\partial}{\partial \tau}$, and then (1.2) reduces to the Nahm equation,

$$\frac{\partial V_a}{\partial \tau} = \frac{1}{2} \epsilon_{abc} [V_b, V_c], \quad (1.6)$$

where V_a ($a = 1, 2, 3$) are volume preserving vector fields on a 3-ball. This form was given by Ashtekar, Jacobson and Smolin [5, 7].

In this paper we present a explicit construction of hyperkähler metrics based on (1.1) and (1.2). We also give a new construction of anti self-dual Yang-Mills connections on hyperkähler manifolds generalizing the multi-instanton Ansatz of 't Hooft, Jackiw-Nohl-Rebbi [9] and further Popov [10, 11, 12].

2 Reductions of the Ashtekar-Mason-Newman equations

We start with the construction of four vector fields V_μ corresponding to Gibbons-Hawking metrics. We use the standard coordinates (x^0, x^1, x^2, x^3) and the volume form $\omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ for the underlying space-time \mathbb{R}^4 . Let ϕ and ψ_i ($i = 1, 2, 3$) be smooth functions and define [13]

$$V_0 = \phi \frac{\partial}{\partial x^0}, \quad (2.1a)$$

$$V_i = \frac{\partial}{\partial x^i} + \psi_i \frac{\partial}{\partial x^0}. \quad (2.1b)$$

Then the volume preserving condition implies that the functions ϕ, ψ_i are independent of x^0 and the half-flat condition implies

$$*d\phi = d\psi, \quad (2.2)$$

where $\psi = \sum_{i=1}^3 \psi_i dx^i$ and $*$ denotes the Hodge operator on the three dimensional sub-space $\mathbb{R}^3 = \{(x^1, x^2, x^3)\}$. These conditions are identical to the Ansatz used by Gibbons-Hawking to construct hyperkähler metrics with a triholomorphic $U(1)$ symmetry [14].

Remark 2.1. The Gibbons-Hawking Ansatz is characterized by the following Lie algebra \mathfrak{g}_{GH} . Let \mathfrak{g}_{GH} be the Lie algebra generated by the Gibbons-Hawking vector fields V_μ . Then \mathfrak{g}_{GH} is given by the extension of the two abelian Lie algebras $\langle \partial_I \phi \frac{\partial}{\partial x^0} \rangle$ and \mathbb{R}^3 . ($\partial_I \phi$ denotes the multiple partial differentiation of ϕ with respect to x^1, x^2 and x^3 .) In other words the following sequence is exact,

$$0 \rightarrow \left\langle \partial_I \phi \frac{\partial}{\partial x^0} \right\rangle \rightarrow \mathfrak{g}_{GH} \rightarrow \mathbb{R}^3 \rightarrow 0. \quad (2.3)$$

We note that $\langle \partial_I \phi \frac{\partial}{\partial x^0} \rangle$ is a left \mathcal{D} -module for

$$\mathcal{D} = \mathbb{R} \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right] / \left(\left(\frac{\partial}{\partial x^1} \right)^2 + \left(\frac{\partial}{\partial x^2} \right)^2 + \left(\frac{\partial}{\partial x^3} \right)^2 \right). \quad (2.4)$$

We now describe an approach which allows us to obtain the vector fields V_μ satisfying the conditions (1.1) and (1.2) from (anti) self-dual Yang-Mills connections of infinite

dimensional gauge groups. Let $\Sigma^{(n)}$ ($n = 1, 2, 3, 4$) be a n dimensional manifold equipped with a volume form $\omega^{(n)}$. Then we assume that the gauge Lie algebra is the Lie algebra $sdiff(\Sigma^{(n)})$ of all volume preserving vector fields on $\Sigma^{(n)}$. Connections on Euclidean space $\mathbb{R}^4 = \{(x^0, x^1, x^2, x^3)\}$ may be explicitly expressed by 1-forms on \mathbb{R}^4 valued in $sdiff(\Sigma^{(n)})$. We write these 1-forms as $A_\mu dx^\mu$ ($\mu = 0, 1, 2, 3$) and require the following conditions:

- (1) A_μ are \mathbb{R}^n -invariant with respect to the coordinates (x^0, \dots, x^{n-1}) .
- (2) A_μ are anti self-dual connections, namely, the covariant differentiations $D_\mu = \frac{\partial}{\partial x^\mu} + A_\mu$ satisfy the equation

$$\frac{1}{2} \bar{\eta}_{\mu\nu}^a [D_\mu, D_\nu] = 0. \quad (2.5)$$

- (3) A_μ ($0 \leq \mu \leq n-1$) are linearly independent at each point on $\Sigma^{(n)}$.

We then define the four vector fields V_μ on $\Sigma^{(n)} \times \mathbb{R}^{4-n}$ as follows:

$$V_\mu = \begin{cases} A_\mu & (0 \leq \mu \leq n-1), \\ D_\mu & (n \leq \mu \leq 3). \end{cases} \quad (2.6)$$

These vector fields evidently preserve the volume form $\omega = \omega^{(n)} \wedge dx^n \wedge \dots \wedge dx^3$ and satisfy the half-flat condition (1.2) and hence induce a hyperkähler metric on $\Sigma^{(n)} \times \mathbb{R}^{4-n}$.

We note the previous Gibbons-Hawking vector fields can be obtained by applying the above construction to the case $\Sigma^{(1)} = \mathbb{R}$. (In the cases of $\Sigma^{(4)}$ and $\Sigma^{(3)}$, we simply recover the equations (1.1), (1.2) and (1.6).)

We now concentrate our attention on the explicit construction of hyperkähler metrics in the case of $X = \Sigma \times \mathbb{R}^2$, where (Σ, ω_Σ) is a two dimensional symplectic manifold. Let us assume that the \mathbb{R}^2 -invariant connections A_μ are the Hamiltonian vector fields X_{f_μ} along $\Sigma \times \{(x^2, x^3)\}$ associated with some functions f_μ ($\mu = 0, 1, 2, 3$) on $\Sigma \times \mathbb{R}^2$. Then (2.6) becomes

$$V_0 = X_{f_0}, \quad (2.7a)$$

$$V_1 = X_{f_1}, \quad (2.7b)$$

$$V_2 = \frac{\partial}{\partial x^2} + X_{f_2}, \quad (2.7c)$$

$$V_3 = \frac{\partial}{\partial x^3} + X_{f_3}. \quad (2.7d)$$

Thus our problem consists in finding the four functions f_μ satisfying the anti self-dual condition (2.5). We will not attempt to answer this question in any generality, but consider three examples below.

- (i) Let f_μ ($\mu = 0, 1, 2$) be x^2 -independent and $f_3 = 0$. Then (2.5) yields the Ward equation [17]:

$$\frac{\partial f_0}{\partial x^3} = \{f_1, f_2\}, \quad \frac{\partial f_1}{\partial x^3} = \{f_2, f_0\}, \quad \frac{\partial f_2}{\partial x^3} = \{f_0, f_1\}, \quad (2.8)$$

where $\{, \}$ denotes the Poisson bracket on Σ induced by the symplectic structure ω_Σ . These equations (but with Poisson brackets replaced by commutators of matrices of some finite dimensional Lie algebra) arose in Nahm's construction of monopole solutions in Yang-Mills theory [15, 16].

The group $SL(2, \mathbb{R})$ acts on $\Sigma = \mathbb{R}^2, H$ (the complex upper half-plane) and $SU(2)$ acts on $\Sigma = S^2$ preserving the standard volume form for each case . So we can construct hyperkähler metrics on $\Sigma \times \mathbb{R} \times I$ (I is an open interval) from the solutions of the Nahm equation valued in $sl(2, \mathbb{R})$ or $su(2)$. For example, we can express such solutions by Jacobi elliptic functions as follows:

(a) $sl(2, \mathbb{R})$

$$f_0 = k \operatorname{sn}(x^3, k) h_0, \quad (2.9a)$$

$$f_1 = k \operatorname{cn}(x^3, k) h_1, \quad (2.9b)$$

$$f_2 = \operatorname{dn}(x^3, k) h_2, \quad (2.9c)$$

(b) $su(2)$

$$f_0 = \operatorname{ns}(x^3, k) \hat{h}_0, \quad (2.10a)$$

$$f_1 = \operatorname{ds}(x^3, k) \hat{h}_1, \quad (2.10b)$$

$$f_2 = \operatorname{cs}(x^3, k) \hat{h}_2, \quad (2.10c)$$

where $k \in \mathbb{R} \setminus \{0\}$ and

$$\begin{aligned} h_0 &= \{h_1, h_2\}, & h_1 &= -\{h_2, h_0\}, & h_2 &= -\{h_0, h_1\}, \\ \hat{h}_0 &= -\{\hat{h}_1, \hat{h}_2\}, & \hat{h}_1 &= -\{\hat{h}_2, \hat{h}_0\}, & \hat{h}_2 &= -\{\hat{h}_0, \hat{h}_1\}. \end{aligned}$$

Some explicit solutions of the Nahm equations valued in finite dimensional Lie algebras were constructed in [17, 18, 19].

(ii) We consider the hyperkähler metric with one rotational Killing symmetry preserving one complex structure but not triholomorphic [20]. We take $\Sigma = \mathbb{R}^2$ with the coordinates (y^0, y^1) and introduce a y^0 -independent function $\psi(y^1, x^2, x^3)$. Let us assume the form,

$$f_0 = -2e^{\frac{\psi}{2}} \cos \frac{y^0}{2}, \quad (2.11a)$$

$$f_1 = 2e^{\frac{\psi}{2}} \sin \frac{y^0}{2}, \quad (2.11b)$$

$$f_2 = -\int^{y^1} \frac{\partial \psi}{\partial x^3} dy^1, \quad (2.11c)$$

$$f_3 = \int^{y^1} \frac{\partial \psi}{\partial x^2} dy^1. \quad (2.11d)$$

Then the functions f_μ satisfy the anti self-dual condition (2.5), if ψ is a solution of three dimensional continual Toda equation:

$$\left(\frac{\partial}{\partial x^2} \right)^2 \psi + \left(\frac{\partial}{\partial x^3} \right)^2 \psi + \left(\frac{\partial}{\partial y^1} \right)^2 e^\psi = 0. \quad (2.12)$$

This solution leads to a hyperkähler metric with the Killing vector field $\frac{\partial}{\partial y^0}$, which is known as the real heaven solution in the Plebański formalism [21].

(iii) Using the solutions of both Nahm and Laplace equations, Popov presented a construction of self-dual Yang-Mills connections on \mathbb{R}^4 [10]. We may apply the same method to our case, i.e. \mathbb{R}^2 -invariant Yang-Mills connections of the gauge Lie algebra $\mathfrak{sdiff}(\Sigma)$. Let $T^a(t)$ ($a = 1, 2, 3$) and $u(x^2, x^3)$ be solutions of the Nahm equation associated with the Lie algebra $\mathfrak{sdiff}(\Sigma)$ and the Laplace equation on \mathbb{R}^2 , respectively:

$$\frac{\partial T^a}{\partial t} = \frac{1}{2} \epsilon_{abc} \{T^b, T^c\} \quad (a, b, c = 1, 2, 3), \quad (2.13a)$$

$$\sum_{\mu=2}^3 \left(\frac{\partial}{\partial x^\mu} \right)^2 u = 0. \quad (2.13b)$$

Then we obtain the solutions f_μ ($\mu = 0, 1, 2, 3$):

$$f_\mu = \sum_{\nu=2}^3 \bar{\eta}_{\mu\nu}^a \frac{\partial u}{\partial x^\nu} T^a(u). \quad (2.14)$$

Remark 2.2. We note that the Ansatz for the vector fields (2.7) is similar to that in [22, 23, 24], which should be regarded as the Ansatz for the vector fields on a complex four dimensional manifold, so that it is not so obvious whether the corresponding metrics satisfy the reality condition. However, our construction manifestly satisfies such a condition.

3 Yang-Mills instantons on hyperkähler manifolds

Now we present a construction of anti self-dual Yang-Mills connections of arbitrary gauge Lie algebra \mathfrak{g} on a hyperkähler manifold X . As mentioned above, Popov have obtained a formula for self-dual Yang-Mills connections on \mathbb{R}^4 . We generalize their construction to the case of four dimensional hyperkähler manifolds by using Ashtekar variables.

Suppose X is a hyperkähler manifold expressed by linearly independent four vector fields V_μ and a volume form ω as mentioned in (1.1) and (1.2).

Proposition 3.1. The \mathfrak{g} -valued connection A on X given by

$$A(V_\mu) = \bar{\eta}_{\mu\nu}^a (V_\nu u) T^a(u) \quad (3.1)$$

satisfies the anti self-dual condition if and only if the set of \mathfrak{g} -valued functions T^a ($a = 1, 2, 3$) on \mathbb{R} is a solution of the Nahm equation:

$$\frac{dT^a}{dt} = \frac{1}{2} \epsilon_{abc} [T^b, T^c], \quad (3.2)$$

and the function u on X is a solution of the equation:

$$\sum_{\mu=0}^3 (V_\mu V_\mu) u = 0. \quad (3.3)$$

Proof Using the identities (1.3), we calculate the anti self-dual part of the curvature $F_{\mu\nu} = F(V_\mu, V_\nu)$ as follows:

$$\begin{aligned} \frac{1}{2}\bar{\eta}_{\mu\nu}^a F_{\mu\nu} &= \frac{1}{2}\bar{\eta}_{\mu\nu}^a \{V_\mu(A(V_\nu)) - V_\nu(A(V_\mu)) - A([V_\mu, V_\nu]) + [A(V_\mu), A(V_\nu)]\} \\ &= -\sum_\sigma (V_\sigma V_\sigma u) T^a - \sum_\sigma (V_\sigma u)(V_\sigma u) \left\{ \frac{dT^a}{dt} - \frac{1}{2}\epsilon_{abc}[T^b, T^c] \right\} \\ &\quad - \frac{1}{2}A(\bar{\eta}_{\mu\nu}^a [V_\mu, V_\nu]) \end{aligned}$$

It follows from (1.2) and our assumptions (3.2) and (3.3) that the above quantity vanishes. \square

Remark 3.2. The equation (3.3) is identical to the Laplace equation associated with the hyperkähler metric. So we can choose the harmonic functions as u in (3.3). For example, if the solutions of (1.2) are the Hamiltonian vector fields for functions f_μ on a symplectic four dimensional manifold, then f_μ are harmonic.

Remark 3.3. In the above proof we don't use the volume preserving condition for the vector fields V_μ . Hence it is not necessary that X is a four dimensional hyperkähler manifold. The restriction for a n dimensional manifold X we require is simply the existence of the linearly independent n vector fields V_μ such that $\frac{1}{2}\bar{\eta}_{\mu\nu}^a [V_\mu, V_\nu] = 0$ with the relations (1.3) for the matrices $\bar{\eta}_{\mu\nu}^a$ [25].

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